Math 564: Advance Analysis 1
Lecture 23

Def. A function $f: \mathbb{R} \to \mathbb{R}$ is called a *distribution* of a loc. fin. Borel measure μ on $\mathbb{R}$ if $\mu((a,b]) = f(b) - f(a)$ for all $a < b$.
Any two distr. functions of the same measure differ by a constant.

We would like to characterize those distribution functions that correspond to $\mu \ll \lambda$. Let f be a distribution of a loc. fin. measure μ on $\mathbb{R}$.
If $\mu \ll \lambda$, then for each (a,b), $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall$ Borel $B \subseteq (a,b)$,

$$\lambda(B) \leq \delta \Rightarrow \mu(B) \leq \varepsilon.$$

When B is open, $B = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n)$, so

$$\lambda(B) = \sum_n (b_n - a_n) \quad \text{and} \quad \mu(B) = \sum_n \mu((a_n, b_n)) \overset{\mu(b_n)=0 \text{ by } \mu \ll \lambda}{=} \sum_n (f(b_n) - f(a_n)),$$

thus,

$$\sum_n (b_n - a_n) \leq \delta \Rightarrow \sum_n f(b_n) - f(a_n) \leq \varepsilon.$$

Def. We say that $f: \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* on (a,b) if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for all open $U \subseteq (a,b)$, writing $U = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n)$, we have

$$\sum_{n \in \mathbb{N}} (b_n - a_n) \leq \delta \Rightarrow \sum_{n \in \mathbb{N}} |f(b_n) - f(a_n)| \leq \varepsilon.$$

We say that f is *locally absolutely continuous* if it is abs. cont. on every (a,b).

Note that if f is abs. cont. on (a,b), then it is unif. cont. on (a,b).

Example. Lipschitz functions are globally absolutely continuous.

So unif. cont $\Leftarrow$ abs. cont. $\Leftarrow$ Lipschitz $\Leftarrow$ having bdd derivative.

Thm. For an increasing $f : \mathbb{R} \to \mathbb{R}$, TFAE:

(1) f is a distribution of a (unique) loc. fin. Borel measure μ on $\mathbb{R}$ s.t. $\mu \ll \lambda$.

(2) FTC holds for f, i.e. f' exists a.e. and $f(b)-f(a) = \int f' d\lambda$ $\forall a<b$.

(3) f is locally abs. continuous.

Proof. We've already proven (1)$\Leftrightarrow$(2) and we argued (1)$\Rightarrow$(3) above.

(3)$\Rightarrow$(1). This is just by the regularity of λ.

Note that f is continuous, so $\exists$ unique atomeless measure μ s.t. f is a distribution of μ. We need to show that if $B \subseteq \mathbb{R}$ is λ-null then B is μ-null. It is enough to show for $B \subseteq (a,b)$. Suppose $\lambda(B)=0$. We will show that $\mu(B) \le \varepsilon$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$ and let $\delta > 0$ be witnessing abs. cont. for this (a,b) and ε. By the regularity of λ, there is an open $B \subseteq U \subseteq (a,b)$ s.t. $\lambda(U) \le \delta$. Then $U = \bigsqcup_n (a_n, b_n)$, so
$$\sum_n (b_n - a_n) = \lambda(U) \le \delta, \text{ therefore}$$

$$\sum_n f(b_n) - f(a_n) \le \varepsilon. \text{ But } \mu(B) \le \mu(U) = \sum_n \mu((a_n,b_n)) =$$
$$= \sum_n \left(\lim_{x \to b_n^+} f(x) - f(a_n)\right) \overset{f \text{ is continuous}}{=} \sum_n f(b_n) - f(a_n) \le \varepsilon. \quad \square$$

What about nonincreasing functions? At best, they would be distributions of signed measures. Because it is annoying to describe distributions of unbdd signed measures, we restrict to finite signed measures on $\mathbb{R}$.

Def. For a signed measure ν, let $\nu = \nu_+ - \nu_-$ be the Hahn decomposition, i.e. ν_+, ν_- are measures. Denote $\nu_* := \nu_+ + \nu_-$ and call it the **total variation** of ν. We'd say that ν is finite if it only takes finite values, equivalently, if ν_* is finite.

We already know that

$$\text{fin. Borel measures on } \mathbb{R} \overset{\text{correspond}}{\longleftrightarrow} \text{bdd increasing right-cont functs.}$$

$$\text{fin. Borel signed measures on } \mathbb{R} \longleftrightarrow \ ?$$

We define a distribution of a Borel signed measure ν as a function $f:\mathbb{R}\to\mathbb{R}$ s.t. $\nu((a,b]) = f(b)-f(a)$.

Let's fixed a fin. Borel signed meas. ν and a distribution f of it, e.g. $f(x) := \nu((-\infty,x])$. Then again f is bdd and right-continuous by the same argument as before. Because of Hahn decomposition, $f = f_{\nu_+} - f_{\nu_-}$, where f_{ν_+}, f_{ν_-} are bdd increasing right-continuous functions. But we'd like a more algorithmic condition instead.

We explore what consequence the finiteness of the total variation ν_* has on f. Let $B := \bigsqcup_{n\in\mathbb{N}} (a_n, b_n]$. Then $\nu_*(B) \le \nu_*(\mathbb{R}) < \infty$.

Thus,

$$\infty > \nu_*(\mathbb{R}) \ge \nu_*(B) = \sum_n \nu_*((a_n,b_n]) \ge \sum_n |\nu((a_n,b_n])| = \sum_n |f(b_n)-f(a_n)|.$$

This motivates the following definition:

Def. For $f:\mathbb{R}\to\mathbb{R}$, let $T_f:\mathbb{R}\to[0,\infty]$ be defined by

$$T_f(x) := \sup\left\{ \sum_{i=0}^{n} |f(x_{i+1})-f(x_i)| : n\in\mathbb{N}, -\infty < x_0 < x_1 < \dots < x_n \le x \right\}.$$

T_f is called the total variation of f and we say that f is of bdd variation if $T_f(\infty) := \lim_{x\to\infty} T_f(x) < \infty$.

Note that $T_f(b)-T_f(a) = \sup\left\{ \sum_{i=0}^{n} |f(x_{i+1})-f(x_i)| : n\in\mathbb{N}, a \le x_0 < \dots < x_n \le b \right\}$

"=" total vertical distance traveled by the graph of $f|_{[a,b]}$.

Thm. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. TFAE:

(1) f is a distribution of a unique fin. Borel signed measure ν.

(2) f is right-continuous and has bdd. variation.

Proof. We have shown (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (1) is outlined in homework, follows from the fact that T_f+f and T_f-f are increasing functions and $f=\frac{1}{2}(T_f+f)-\frac{1}{2}(T_f-f)$. $\square$

One can also prove the analogue of the FTC:

Thm. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. TFAE:

(1) f is a distr. of a unique fin. Borel signed meas. s.t. $\nu_f \ll \lambda$.

(2) f' exists a.e. and is in $L^1(\lambda)$; moreover, $f(b)-f(a)=\int_a^b f'\,d\lambda$

(3) f has bdd variation and is abs. continuous.

Proof. This follows from the analogous thm for increasing functions and the fact that bdd var $\Leftrightarrow$ being a difference of two bdd increasing functions. $\square$

Normed vector spaces and L^p spaces.

Normed vector spaces. A norm on a (real) vector space X is a function $\|\cdot\|: X\to[0,\infty)$ s.t.

(i) $x=0 \Leftrightarrow \|x\|=0$.

(ii) $\|c\cdot x\| = |c|\cdot\|x\|$

(iii) $\|x+y\|\le\|x\|+\|y\|$.

When in (i) only $\Rightarrow$ holds, we call it a pseudo-norm or seminorm.

A vector space X equipped with a norm is called a normed vector space.

In a normed vector space X, the norm defines a metric $d(x,y) := \|x-y\|$, which we call the norm metric. The topology induced by this metric is called the norm topology of X.
Thinking of X as a metric space, it makes sense to talk about its completeness, i.e. every Cauchy sequence converges. The following is a convenient characterization of completeness for normed vector spaces.

Def. For a sequence $(x_n) \subseteq X$, $(X, \|\cdot\|)$ normed vector space, we say that the series $\sum_{n\in\mathbb{N}} x_n$ converges, if $\left(\sum_{i=0}^{n} x_i\right)_n$ is convergent in norm. We denote its limit by $\sum_{n\in\mathbb{N}} x_n$. We say that the series $\sum_{n\in\mathbb{N}} x_n$ is absolutely convergent if $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.

Characterization of completeness. A normed vector space is complete if and only if every absolutely convergent series converges (in norm).